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# KMS States for generalized Gauge actions on Cuntz-Krieger algebras

## (An application of the Ruelle-Perron-Frobenius Theorem)

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**Abstract.** Given a zero-one matrix $A$ we consider certain one-parameter groups of automorphisms of the Cuntz-Krieger algebra $\mathcal{O}_A$, generalizing the usual gauge group, and depending on a positive continuous function $H$ defined on the Markov space $\Sigma_A$. The main result consists of an application of Ruelle's Perron-Frobenius Theorem to show that these automorphism groups admit a single KMS state.

**Keywords:** C*-algebras, Cuntz-Krieger algebras, KMS states, gauge action, Ruelle-Perron-Frobenius Theorem.

**Mathematical subject classification:** 46L55, 37A55.

## 1 Introduction

In 1978 Olesen and Pedersen [9] showed that the periodic gauge action on the Cuntz algebra $\mathcal{O}_n$ admits a unique KMS state, whose inverse temperature is $\beta = \log n$. Two years later Evans [4: 2.2] extended their result to include, among other things, non-periodic gauge actions, namely one-parameter automorphism groups on $\mathcal{O}_n$ given on the standard generating partial isometries $S_j$ by

$$\gamma_t(S_j) = N_j^{it} S_j, \quad \forall\, t \in \mathbf{R},$$

where $\{N_j\}_{j=1}^n$ is a collection of real numbers with $N_j > 1$ for all $j$. See also [3: 3.1]. In 1984 Enomoto, Fujii and Watatani treated the case of the periodic gauge action on the Cuntz-Krieger algebra $\mathcal{O}_A$ for an irreducible matrix $A$ and again arrived at the conclusion that there exists a unique KMS state. The case

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of a non-periodic gauge action on $\mathcal{O}_A$ was discussed in [8] in the context of Cuntz-Krieger algebras for infinite matrices but, specializing the conclusions to the finite case, one gets the expected result that if the matrix $A$ is irreducible and the parameters $N_j$ are all greater than 1 then there exists a unique KMS state.

The present work aims to take a new step in the direction of understanding the KMS states on Cuntz-Krieger algebras (over finite matrices) by studying generalized gauge actions on $\mathcal{O}_A$. In order to describe these actions let $\Sigma_A$ be the one-sided Markov space for the given matrix $A$ and consider the copy of $C(\Sigma_A)$ within $\mathcal{O}_A$ that is generated by the elements of the form $S_{i_1} \ldots S_{i_k} S_{i_k}^* \ldots S_{i_1}^*$, where the $S_i$ are the standard generating partial isometries. Fixing an invertible element $U \in C(\Sigma_A)$ it is not hard to see that the correspondence

$$S_j \mapsto U S_j$$

extends to give an automorphism of $\mathcal{O}_A$. Therefore if $H \in C(\Sigma_A)$ is a strictly positive element there exists a unique one-parameter automorphism group $\{\gamma_t\}_{t \in \mathbf{R}}$ of $\mathcal{O}_A$ such that

$$\gamma_t(S_j) = H^{it} S_j.$$

We will refer to $\gamma$ as the *generalized gauge action*. It is easy to see that this in fact generalizes both the periodic and the non-periodic gauge actions referred to above.

The goal of this paper, as the title suggests, is to study the KMS states for the generalized gauge action on $\mathcal{O}_A$. Our main result, Theorem 4.4, states that under certain hypotheses there exists a single such KMS state.

The method employed consists of considering $\mathcal{O}_A$ as the crossed product of $C(\Sigma_A)$ by the endomorphism induced by the Markov subshift [6] and applying Theorem 9.6 from [7] to reduce the problem to the search for probability measures on $\Sigma_A$ which are fixed by Ruelle's transfer operator [12, 13, 1, 2]. This turns out to be closely related to Ruelle's version of the Perron-Frobenius Theorem (see e.g. [2: 1.7]), except that the latter deals with eigenvalues for the transfer operator while we need actual fixed points. With not too much effort we are then able to exploit Ruelle's Theorem in order to understand the required fixed points and thus reach our conclusion.

It should be stressed that Ruelle's Theorem requires two crucial hypotheses, namely that the matrix $A$ be *irreducible and aperiodic* in the sense that there exists a positive integer $m$ such that all entries of $A^m$ are strictly positive (see e.g. [1: Section 1.2]), and that $H$ is Hölder continuous. We are therefore forced to postulate these conditions leaving open the question as to whether one could do without them.

The organization of this paper is as follows: in section (2), the longer and more technical section of this work, we give a brief account of Ruelle's Theorem and draw the conclusions we need with respect to the existence and uniqueness of probability measures that are fixed under the transfer operator.

Section (3) is devoted to reviewing results about crossed products by endomorphisms and in the final section we put all the pieces together proving our main result.

After this article circulated as a preprint we learned of Renault's interesting paper [11] on cocycles for AF-equivalence relations which is closely related to what we do here.

I would finally like to acknowledge helpful conversations with M. Viana who, among other things, brought Ruelle's Theorem to my attention.

## 2 Ruelle's Perron-Frobenius Theorem

Beyond establishing our notation this section is intended to present Ruelle's Perron-Frobenius Theorem and to develop some further consequences of it to be used in later sections.

Fix, once and for all, an $n \times n$ matrix $A = \{A_{i,j}\}_{1 \le i,j \le n}$, with $A_{i,j} \in \{0, 1\}$ for all $i$ and $j$, such that no row or column of $A$ is identically zero.

Throughout this paper we will be concerned with the associated (one-sided) subshift of finite type, namely the dynamical system $(\sigma, \Sigma_A)$, where $\Sigma_A$ is the compact topological subspace of the infinite product space $\prod_{i \in \mathbf{N}} \{1, 2, \ldots, n\}$ given by

$$\Sigma_A = \Big\{ x = (x_0, x_1, x_2, \ldots) \in \prod_{i \in \mathbf{N}} \{1, 2, \ldots, n\} : A_{x_i, x_{i+1}} = 1 \text{ for all } i \ge 0 \Big\},$$

and $\sigma : \Sigma_A \to \Sigma_A$ is the "left shift", namely the continuous function given by

$$\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

From the assumption that no column of $A$ is identically zero it follows that $\sigma$ is surjective.

Given a real number $\beta \in (0, 1)$ define a metric $d$ on $\Sigma_A$ by setting

$$d(x, y) = \beta^{N(x,y)}, \quad \forall\, x, y \in \Sigma_A,$$

where $N(x, y)$ is the largest integer $N$ such that $x_i = y_i$ for all $i < N$. In the special case in which $x = y$ we set $N(x, y) = +\infty$ and interpret $\beta^{N(x,y)}$ as being zero. It is easy to see that this metric is compatible with the product topology.

Let $C(\Sigma_A)$ denote the C*-algebra of all continuous complex functions on $\Sigma_A$. We will consider the operator

$$\mathcal{L} : C(\Sigma_A) \to C(\Sigma_A)$$

given by

$$\mathcal{L}(f)\big|_x = \sum_{y\in\sigma^{-1}(\{x\})} f(y), \quad \forall f \in C(\Sigma_A), \quad \forall x \in \Sigma_A. \tag{2.1}$$

Since $\sigma$ is surjective one has that $\sigma^{-1}(\{x\})$ is never empty. It is also clear that $\sigma^{-1}(\{x\})$ has at most $n$ elements so that the above sum is finite for every $x$. One checks that $\mathcal{L}(f)$ is indeed a continuous function and hence that $\mathcal{L}$ is a well defined linear operator on $C(\Sigma_A)$, which is moreover positive and bounded.

Given a real continuous function $\phi$ on $\Sigma_A$ the operator

$$\mathcal{L}_\phi : C(\Sigma_A) \to C(\Sigma_A)$$

given by $\mathcal{L}_\phi(f) = \mathcal{L}(e^\phi f)$ was introduced by Ruelle in [12: 2.3] (see also [13], [2], and [1]) and it is usually referred to as *Ruelle's transfer operator*.

Most of the time we will assume that $\phi$ is Hölder continuous with respect to the metric $d$ above: recall that a complex function $\phi$ on a metric space $M$ is said to be *Hölder continuous* when one can find positive constants $K$ and $\alpha$ such that $|\phi(x) - \phi(x)| \le K d(x, y)^\alpha$, for all $x$ and $y$ in $M$.

The most important technical tool to be used in this work is the celebrated Ruelle-Perron-Frobenius Theorem which we now state for the convenience of the reader.

**Theorem 2.2.** (D. Ruelle) *Let $A$ be an $n \times n$ zero-one matrix and let $\phi$ be a real function defined on $\Sigma_A$. Suppose that:*

(a) *There exists a positive integer $m$ such that $A^m > 0$ (in the sense that all entries are $> 0$), and*

(b) *$\phi$ is Hölder continuous.*

*Then there are: a strictly positive function $h \in C(\Sigma_A)$, a Borel probability measure $\nu$ on $\Sigma_A$, and a real number $\lambda > 0$, such that*

(i) $\mathcal{L}_\phi(h) = \lambda h$,

(ii) $\mathcal{L}_\phi^*(\nu) = \lambda\nu$, *where $\mathcal{L}_\phi^*$ is the adjoint operator acting on the dual of $C(\Sigma_A)$, and*

(iii) *for every $g \in C(\Sigma_A)$ one has that* $\lim\limits_{k\to\infty} \|\lambda^{-k}\mathcal{L}_\phi^k(g) - \nu(g)h\| = 0$.

**Proof.** See e.g. [2: 1.7].

**Proposition 2.3.** *Under the hypotheses of (2.2) there exists a unique pair* $(\lambda_1, \nu_1)$ *such that* $\lambda_1$ *is a complex number,* $\nu_1$ *is a probability measure on* $\Sigma_A$*, and* $\mathcal{L}_\phi^*(\nu_1) = \lambda_1 \nu_1$.

**Proof.** The existence obviously follows from (2.2.ii). As for uniqueness let $(\lambda_1, \nu_1)$ be such a pair and let $(\lambda, \nu)$ be as in (2.2). For all $g \in C(\Sigma_A)$ we have

$$\lim_{k\to\infty} \left(\frac{\lambda_1}{\lambda}\right)^k \nu_1(g) = \lim_{k\to\infty} \nu_1\left(\lambda^{-k}\mathcal{L}_\phi^k(g)\right) = \nu(g)\nu_1(h),$$

by (2.2.iii). Plugging $g = 1$ above we conclude that the sequence $\left(\frac{\lambda_1}{\lambda}\right)^k$ converges to the nonzero value $\nu_1(h)$ but this is only possible if $\lambda_1 = \lambda$. For every $g$ we then have that $\nu_1(g) = \nu(g)\nu_1(h)$, so $\nu_1$ is proportional to $\nu$. But since these are probability measures we must have $\nu_1 = \nu$. □

In particular it follows that both the $\lambda$ and the $\nu$ in the conclusion of (2.2) are uniquely determined. In the following we give an explicit way to compute $\lambda$ in terms of $\mathcal{L}_\phi$ (see [1: 1.39]).

**Proposition 2.4.** *Under the hypotheses of (2.2) one has that*

$$\lambda = \lim_{k\to\infty} \|\mathcal{L}_\phi^k(1)\|^{1/k}.$$

**Proof.** Plugging $g = 1$ in (2.2.iii) we conclude that

$$\lim_{k\to\infty} \lambda^{-k}\|\mathcal{L}_\phi^k(1)\| = \|h\| > 0.$$

So we may choose $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$

$$\frac{\|h\|}{2} < \lambda^{-k}\|\mathcal{L}_\phi^k(1)\| < 2\|h\|.$$

Taking $k^{th}$ roots and then the limit as $k \to \infty$ we get the conclusion. □

In the application of Ruelle's Theorem that we have in mind we will take

$$\phi = \phi_\beta = -\beta \log(H), \tag{2.5}$$

where $H$ is a strictly positive continuous function on $\Sigma_A$ and $\beta > 0$ is a real number.

Observe that if $H$ is Hölder continuous then so is $\phi_\beta$ for every real $\beta$ (this is because "log" is Lipschitz on every compact subset of $(0, +\infty)$, e.g. the range of $H$). In this case Ruelle's Theorem gives a correspondence $\beta \to \lambda$ which we would like to explore more closely in what follows.

**Proposition 2.6.** *Let $A$ be an $n \times n$ zero-one matrix satisfying (2.2.a) and suppose that $H$ is a Hölder continuous function on $\Sigma_A$ such that*

$$H(y) > 1, \quad \forall\, y \in \Sigma_A.$$

*For every $\beta \geq 0$ let $\phi_\beta$ be as in (2.5) and denote by $\lambda(\beta)$ the unique $\lambda$ satisfying the conditions of (2.2) for $\phi = \phi_\beta$. Then one has that*

(i) $\lambda(0) > 1$,

(ii) $\lim\limits_{\beta \to \infty} \lambda(\beta) = 0$, *and*

(iii) $\lambda$ *is a strictly decreasing continuous function of* $\beta$.

**Proof.** Observe that

$$\mathcal{L}_{\phi_\beta}(f) = \mathcal{L}(e^{\phi_\beta} f) = \mathcal{L}\left(e^{-\beta \log(H)} f\right) = \mathcal{L}\left(H^{-\beta} f\right).$$

Let $m$ and $M$ be the supremum and infimum of $H$ on $\Sigma_A$, respectively. For every $\beta \geq 0$ and $y \in \Sigma_A$ one therefore has that

$$M^{-\beta} \leq H(y)^{-\beta} \leq m^{-\beta},$$

so that if $f \in C(\Sigma_A)$ is nonnegative we have

$$M^{-\beta} \mathcal{L}(f) \leq \mathcal{L}_{\phi_\beta}(f) \leq m^{-\beta} \mathcal{L}(f).$$

By induction it is easy to see that for all $k \in \mathbf{N}$

$$M^{-k\beta} \mathcal{L}^k(f) \leq \mathcal{L}^k_{\phi_\beta}(f) \leq m^{-k\beta} \mathcal{L}^k(f).$$

Taking norms and $k^{th}$ roots we conclude that

$$M^{-\beta} \|\mathcal{L}^k(f)\|^{1/k} \leq \|\mathcal{L}^k_{\phi_\beta}(f)\|^{1/k} \leq m^{-\beta} \|\mathcal{L}^k(f)\|^{1/k}.$$

Plugging $f = 1$ above and observing that $1 \leq \|\mathcal{L}^k(1)\| \leq \|\mathcal{L}\|^k$ we obtain

$$M^{-\beta} \leq \|\mathcal{L}^k_{\phi_\beta}(1)\|^{1/k} \leq m^{-\beta}\|\mathcal{L}\|,$$

and hence (2.4) yields

$$M^{-\beta} \leq \lambda(\beta) \leq m^{-\beta}\|\mathcal{L}\|.$$

Observing that $H > 1$, and hence that $m > 1$, we deduce (ii). It is also clear from the above that $\lambda(0) \geq 1$ so it is enough to show that $\lambda(0) \neq 1$ in order to obtain (i).

Arguing by contradiction suppose that $\lambda(0) = 1$. Let $h > 0$ be given by (2.2) so that $\mathcal{L}_{\phi_0}(h) = \mathcal{L}(h) = h$. Choose $x^0 \in \Sigma_A$ such that $h(x^0) = \inf_{y\in\Sigma_A} h(y)$, and observe that, since

$$h(x^0) = \sum_{y\in\sigma^{-1}(\{x^0\})} h(y),$$

there exists a unique $y$ in $\sigma^{-1}(\{x^0\})$ which moreover satisfies $h(y) = h(x^0)$. Repeating this process one obtains a sequence $\{x^k\}_{k\in\mathbf{N}}$ in $\Sigma_A$ such that $\sigma^{-1}(\{x^k\}) = \{x^{k+1}\}$ for all $k$. Letting $x_k = x^k_0$ (the zero$^{th}$ coordinate of $x^k$) we have that $A_{x_{k+1},x_k} = 1$ and also that this is the only nonzero entry of $A$ in the column $x_k$. Since $A$ is a finite matrix the sequence $\{x_k\}$ must be periodic. Assuming without loss of generality that the first period of this sequence is $\{1, \ldots, m\}$, where $m \leq n$, we see that $A$ has the form

$$A = \begin{bmatrix} S_m & B \\ 0 & C \end{bmatrix},$$

where $S_m$ is the matrix of the forward permutation of $m$ elements. However this is easily seen to contradict (2.2.a) both when $m < n$ (because the zero block in the lower left corner will appear in any power of $A$) and when $m = n$ (because $S_m$ definitely fails to satisfy (2.2.a)).

In order to prove (iii) let $\delta > 0$ so that $m^\delta \leq H(y)^\delta \leq M^\delta$ for all $y$ in $\Sigma_A$. Given $\beta \in \mathbf{R}$ we then have that

$$m^\delta H(y)^{-\beta} \leq H(y)^{-(\beta-\delta)} \leq M^\delta H(y)^{-\beta}.$$

For every nonnegative continuous function $f$ it follows that

$$m^\delta \mathcal{L}_{\phi_\beta}(f) \leq \mathcal{L}_{\phi_{\beta-\delta}}(f) \leq M^\delta \mathcal{L}_{\phi_\beta}(f),$$

and if $k \in \mathbf{N}$ one has

$$m^{k\delta} \mathcal{L}^k_{\phi_\beta}(f) \leq \mathcal{L}^k_{\phi_{\beta-\delta}}(f) \leq M^{k\delta} \mathcal{L}^k_{\phi_\beta}(f).$$

Taking norms and $k^{th}$ roots we conclude that

$$m^\delta \|\mathcal{L}^k_{\phi_\beta}(f)\|^{1/k} \leq \|\mathcal{L}^k_{\phi_{\beta-\delta}}(f)\|^{1/k} \leq M^\delta \|\mathcal{L}^k_{\phi_\beta}(f)\|^{1/k}.$$

With $f = 1$ and taking the limit as $k \to \infty$, we get by (2.4) that

$$m^\delta \lambda(\beta) \leq \lambda(\beta - \delta) \leq M^\delta \lambda(\beta). \tag{2.7}$$

Substituting $\beta + \delta$ for $\beta$ above leads to

$$M^{-\delta} \lambda(\beta) \leq \lambda(\beta + \delta) \leq m^{-\delta} \lambda(\beta). \tag{2.8}$$

By (2.7) and (2.8) one sees that $\lambda$ is a continuous function of $\beta$. Since $m > 1$ by hypothesis the rightmost inequality in (2.8) gives $\lambda(\beta + \delta) < \lambda(\beta)$ and hence that $\lambda$ is strictly decreasing. ◻

**Corollary 2.9.** *Under the hypotheses of (2.6) there exists a unique $\beta > 0$ such that $\lambda(\beta) = 1$.*

## 3 Preliminaries on Crossed Products

Define the map $\alpha : C(\Sigma_A) \to C(\Sigma_A)$ by the formula

$$\alpha(f) = f \circ \sigma, \quad \forall f \in C(\Sigma_A).$$

It is easy to see that $\alpha$ is a C*-algebra endomorphism of $C(\Sigma_A)$. Since $\sigma$ is surjective one has that $\alpha$ is injective. We should also notice that $\alpha(1) = 1$.

For $x \in \Sigma_A$ let

$$Q(x) = \#\big\{y \in X : \sigma(y) = x\big\},$$

("#" meaning number of elements). Alternatively $Q(x)$ may be defined as the number of "ones" in the column of $A$ indexed by $x_0$. Therefore $1 \leq Q(x) \leq n$ for all $x \in \Sigma_A$ so that in particular $Q$ is invertible as an element of $C(\Sigma_A)$. Define the operator

$$\mathcal{L} : C(\Sigma_A) \to C(\Sigma_A)$$

by $\mathcal{L}(f) = Q^{-1}\mathscr{L}(f)$, where $\mathscr{L}$ is defined in (2.1). It is easy to see that $Q = \mathscr{L}(1)$ and hence that $\mathcal{L}(1) = 1$. Moreover

$$\mathcal{L}\big(\alpha(f)g\big) = f\mathcal{L}(g), \quad \forall f, g \in C(\Sigma_A),$$

which tells us that $\mathcal{L}$ is a *transfer operator* for the pair $(C(\Sigma_A), \alpha)$ according to Definition (2.1) in [6]. One may therefore construct the crossed product algebra

$$C(\Sigma_A) \rtimes_{\alpha,\mathcal{L}} \mathbf{N},$$

or $C(\Sigma_A) \rtimes \mathbf{N}$, for short, as in [6:3.7], which turns out to be a C*-algebra generated by a copy of $C(\Sigma_A)$ and an extra element $S$ which, among other things, satisfies

- $S^*S = 1$,
- $Sf = \alpha(f)S$, and
- $S^*fS = \mathcal{L}(f)$,

for all $f \in C(\Sigma_A)$. See [6] for the precise definition of $C(\Sigma_A) \rtimes \mathbf{N}$.

In [6:6.2] it is proved that $C(\Sigma_A) \rtimes \mathbf{N}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$. It will be convenient for us to bear in mind the isomorphism between $\mathcal{O}_A$ and $C(\Sigma_A) \rtimes \mathbf{N}$ given in [6], which we next describe. For this consider for each $j = 1, \ldots, n$, the clopen subset $\Sigma^j$ of $\Sigma_A$ given by

$$\Sigma^j = \{x \in \Sigma_A : x_0 = j\}.$$

These are precisely the sets forming the standard Markov partition of $\Sigma_A$. Also let $P_j$ be the characteristic function of $\Sigma^j$. According to [6] there exists an isomorphism

$$\Psi : \mathcal{O}_A \to C(\Sigma_A) \rtimes \mathbf{N}$$

which is determined by the fact that the canonical generating partial isometries $S_j \in \mathcal{O}_A$ are mapped under $\Psi$ as follows:

$$\Psi(S_j) = P_j S Q^{1/2}.$$

We would next like to review the definition of the generalized gauge action on $\mathcal{O}_A$. For this fix a strictly positive element $H \in C(\Sigma_A)$. According to [7:6.2] there exists a unique one parameter automorphism group $\gamma$ of $C(\Sigma_A) \rtimes \mathbf{N}$ such that for all $t \in \mathbf{R}$,

$$\gamma_t(S) = H^{it}S, \quad \text{and} \quad \gamma_t(f) = f, \quad \forall\, f \in C(\Sigma_A).$$

Transferring $\gamma$ to $\mathcal{O}_A$ via the isomorphism $\Psi$ described above one gets an automorphism group on $\mathcal{O}_A$ which is characterized by the fact that

$$\gamma_t(S_j) = H^{it}S_j, \quad \forall\, j = 1, \ldots, n.$$

Observe that in case $H$ is a constant function, say everywhere equal to Neper's number $e$, and $A_{ij} \equiv 1$, then $\mathcal{O}_A$ coincides with the Cuntz algebra $\mathcal{O}_n$ and one recovers the action over $\mathcal{O}_n$ considered in [9]. We shall refer to this as the *scalar gauge action*.

For a slightly more general example suppose that $H$ is constant on each $\Sigma^j$, taking the value $N_j$ there. Then

$$\gamma_t(S_j) = H^{it} S_j = H^{it} P_j S_j = N_j^{it} S_j,$$

and we obtain special cases of actions studied in [4] or [8].

Observe that the composition $E = \alpha \circ \mathcal{L}$ is a conditional expectation from $C(\Sigma_A)$ onto the range of $\alpha$. By [7: Section 11], using the set $\{P_1, \ldots, P_n\}$, we see that $E$ is of index-finite type. It therefore follows from [7:8.9] that there exists a unique conditional expectation

$$G : \mathcal{O}_A \to C(\Sigma_A)$$

which is invariant under the scalar gauge action. This conditional expectation must therefore coincide with the conditional expectation given by [5:2.9] for the Cuntz-Krieger bundle (see [10] and [5]).

Let us now give a concrete description of $G$ based on the well known fact that $\mathcal{O}_A$ is linearly spanned by the set of all $S_\mu S_\nu^*$, where $\mu$ and $\nu$ are finite words in the alphabet $\{1, \ldots, n\}$, and we let $S_\mu = S_{\mu_0} \ldots S_{\mu_k}$ whenever $\mu = \mu_0 \ldots \mu_k$.

For any such $\mu$ and $\nu$ we have by [5] that

$$G(S_\mu S_\nu^*) = \begin{cases} S_\mu S_\nu^* & , \text{ if } \mu = \nu, \\ 0 & , \text{ if } \mu \neq \nu \end{cases} \tag{3.1}$$

## 4 KMS states

It is our main goal to describe the KMS states on $\mathcal{O}_A$ for the gauge action $\gamma$ determined by a given $H$ as above. Recall from [7:9.6] that for every $\beta > 0$ the correspondence

$$\psi \mapsto \nu = \psi|_{C(\Sigma_A)} \tag{4.1}$$

is a bijection from the set of KMS$_\beta$ states $\psi$ on $C(\Sigma_A) \rtimes \mathbf{N}$ and the set of probability measures[1] $\nu$ on $\Sigma_A$ such that

$$\nu(f) = \nu\Big(\mathcal{L}\big(H^{-\beta}\text{ind}(E) f\big)\Big), \quad \forall f \in C(\Sigma_A), \tag{4.2}$$

[1] By the Riesz Representation Theorem we identify probability measures and states as usual.

where $\mathrm{ind}(E)$ is the Jones-Kosaki-Watatani index of $E$. See [7] for details. As observed in [7:Section 11] the right hand side of (4.2) coincides with $\nu\big(\mathcal{L}_{\phi_\beta}(f)\big)$, where $\phi_\beta$ is as in (2.5), so that (4.2) is equivalent to

$$\mathcal{L}^*_{\phi_\beta}(\nu) = \nu. \tag{4.3}$$

We now arrive at our main result.

**Theorem 4.4.** *Let $A$ be an $n \times n$ zero-one matrix satisfying (2.2.a) and let $H$ be a Hölder continuous function on $\Sigma_A$ such that $H(y) > 1$ for all $y$ in $\Sigma_A$. Let $\gamma$ be the unique one-parameter automorphism group of $\mathcal{O}_A$ such that*

$$\gamma_t(S_j) = H^{it} S_j, \quad \forall\, j = 1, \ldots, n,$$

*where the $S_j$ are the canonical partial isometries generating $\mathcal{O}_A$. Then $\mathcal{O}_A$ admits a unique KMS state $\psi$ for $\gamma$. The inverse temperature at which this state occurs is the unique value of $\beta$ for which $\lambda(\beta) = 1$ (see 2.9). In addition $\psi$ is given by*

$$\psi = \nu \circ G,$$

*where $G$ is the conditional expectation described in (3.1) and $\nu$ is the unique measure on $\Sigma_A$ satisfying the Ruelle-Perron-Frobenius Theorem for $\phi = -\beta \log(H)$. Finally there are no ground states for $\gamma$.*

**Proof.** By (2.9) let $\beta > 0$ be such that $\lambda(\beta) = 1$. Applying (2.2) for $\phi = \phi_\beta = -\beta \log(H)$, let $\nu$ be the unique probability measure on $\Sigma_A$ satisfying

$$\mathcal{L}^*_{\phi_\beta}(\nu) = \lambda(\beta)\nu = \nu.$$

Then (4.3) holds and hence by [7:9.6] the composition $\psi = \nu \circ G$ is a $\mathrm{KMS}_\beta$ state for $\gamma$.

Suppose now that $\beta_1 > 0$ and let $\psi_1$ be a $\mathrm{KMS}_{\beta_1}$ state for $\gamma$. Set $\nu_1 = \psi_1|_{C(\Sigma_A)}$ and observe that, again by [7:9.6], one has that $\nu_1$ satisfies (4.3) for $\phi_{\beta_1}$. So the pair $(1, \nu_1)$ satisfies the conditions of (2.3) and hence $\lambda(\beta_1) = 1$ so that $\beta_1 = \beta$ by (2.9). Also by (2.3) $\nu_1$ must coincide with $\nu$ and hence $\psi_1 = \psi$ because the correspondence in (4.1) is bijective.

That no ground states exist follows from [7:10.1]. □

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